Exercise 18

In Exercises 15-20, you are asked to solve the wave equation (1) subject to the boundary conditions (3), (4), and the initial conditions (5), (6), for the given data. [Hint: Use (10) and the remark that follows it.]

$$f(x) = 0, \quad g(x) = \sin\frac{\pi x}{L}$$

Solution

The general solution to the wave equation on a finite interval with fixed ends and zero shape and arbitrary initial velocity,

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \ -\infty < t < \infty \\
&u(x,0) = 0 \\
&\frac{\partial u}{\partial t}(x,0) = \sin\frac{\pi x}{L} \\
&u(0,t) = 0 \\
&u(L,t) = 0,
\end{aligned}$$

is (to be derived in later chapters)

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}.$$

Start by differentiating u with respect to t.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L} \\
&= \sum_{n=1}^{\infty}\frac{\partial}{\partial t}\left(B_n \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}\right) \\
&= \sum_{n=1}^{\infty}\frac{n\pi c}{L}B_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}
\end{aligned}$$

To determine the constants B_n, set $t = 0$ and substitute the given function for $\frac{\partial u}{\partial t}(x,0)$.

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,0) &= \sum_{n=1}^{\infty}\frac{n\pi c}{L}B_n \sin\frac{n\pi x}{L} \\
\sin\frac{\pi x}{L} &= \frac{\pi c}{L}B_1\sin\frac{\pi x}{L} + \frac{2\pi c}{L}B_2\sin\frac{2\pi x}{L} + \frac{3\pi c}{L}B_3\sin\frac{3\pi x}{L} + \cdots
\end{aligned}$$

Then match the coefficients on both sides.

$$\frac{\pi c}{L}B_1 = 1 \quad \rightarrow \quad B_1 = \frac{L}{\pi c}$$

$$\frac{2\pi c}{L}B_2 = 0$$

$$\vdots$$

$$\frac{n\pi c}{L}B_n = 0, \quad n \neq 1$$

Therefore, the general solution that satisfies the initial conditions is

$$\begin{aligned} u(x,t) &= B_1 \sin\frac{\pi x}{L} \sin\frac{\pi c t}{L} \\ &= \frac{L}{\pi c} \sin\frac{\pi x}{L} \sin\frac{\pi c t}{L}. \end{aligned}$$